II. Final-Stage Decay Law

When $4\nu t/a = t/t_d \gg 1$, and k is not close to 1 [i.e., $t_d/t_g \sim 0(1)$], asymptotic expansions of Eqs. (11) and (12) for large t lead to the following result. Putting $a/4\nu t = \beta$, $ka/4\nu t = \alpha$, and $\alpha = k\beta$, one obtains, from Eq. (11)

$$r_c^2 = \frac{a}{\beta}\frac{(1+\alpha)(1+\beta)}{(\alpha-\beta)}\ln\left(\frac{1+\alpha}{1+\beta}\right) = \frac{a}{\beta}\left[1 + (k+1)\frac{\beta}{2} + \ldots\right]$$

i.e.,

$$r_c = (4\nu t)^{1/2} \tag{13}$$

From Eq. (12), one obtains

$$E = \frac{\rho A^2}{8\pi}\left[2\ln\left(1 + \frac{1+\alpha}{1+\beta}\right) - \ln\left(\frac{1+\alpha}{1+\beta}\right) - \ln 4\right]$$

i.e.,

$$E = \frac{\rho A^2}{32\pi}(k-1)^2\left(\frac{a}{4\nu t}\right)^2 \tag{14}$$

Expressions (13) and (14) constitute the two asymptotic laws that govern the final-stage vortex-core growth and kinetic-energy decay, i.e., "the radius of the vortex core grows in proportion to the square root of the decay time; the content of kinetic energy decays according to the inverse square of the decay time."

It might be of interest to note that the vortex-core radius r_c in Eq. (13) coincides with Taylor's dissipation eddy scale λ of a homogenous turbulence field, although they certainly follow different decay laws. That λ must correspond to a $(-\frac{5}{2})$-power energy-decay law is evident from the dissipation equation

$$\partial u'^2/\partial t = -10\nu u'^2/\lambda^2 \tag{15}$$

The significance of the coincidence, however, will be discussed in a future report.

The inverse-square law for final-stage decay of a viscous vortex [Eq. (14)] follows strictly from the limited time of generation and finite amount of field-energy content. It is not difficult to show that this decay law differs essentially from the decay law corresponding to Lamb's model of viscous decay of a potential vortex field. Indeed, for Lamb's model, the following result was obtained:[3]

$$v = (\Gamma_0/2\pi r)[1 - \exp(-r^2/4\nu t)] \tag{16}$$

Comparing Eqs. (16) and (5), it is at once clear that Lamb's model of vortex decay implies infinite generation time and infinite flow-field kinetic energy.

It might not be out of place at this point to remark that the present single-vortex final-stage energy-decay law, i.e., Eq. (14), played an essential role in the formulation of a recent theory on the final-stage decay of grid-produced turbulence.[4,5]

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Free Vibration of Rectangular and Circular Orthotropic Plates

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Nomenclature

A_1,A_2,A_3 A_4,A_5	=	constants
a,b	=	dimensions of plate
D_x,D_y	=	flexural rigidities of orthotropic plate
c	=	radius of circular plate
D	=	flexural rigidity of isotropic plate
H	=	torsional rigidity of orthotropic plate
h	=	plate thickness
k	=	damping coefficient
r,θ	=	polar coordinates
t	=	time
$w(x,y,t)$	=	deflection of plate
x,y	=	rectangular coordinates
ω	=	circular frequency
ρ	=	mass density of plate material
ν	=	Poisson's ratio
$\nabla_0{}^4$	=	$(D_x\partial^4/\partial x^4) + (2H\partial^4/\partial x^2\partial y^2) + (D_y\partial^4/\partial y^4)$
$\nabla_{11}{}^4\psi$	=	$D_x\psi_{,rrrr} + 2H[(2/r^3)\psi_{,r} - (2/r^2)\psi_{,rr} + (1/r)\psi_{,rrr}] + D_y[-(3/r^3)\psi_{,r} + (3/r^2)\psi_{,rr}]$

Subscripts

t,tt,r,rr, etc. = derivatives with respect to t and r, respectively

The frequencies of the fundamental normal modes of free vibration for rectangular and circular orthotropic plates are obtained by using the Galerkin method as formulated by Stanišić.[1] The results are compared with values obtained by other methods. Although vibration analysis of rectangular orthotropic plates has been considered by many writers,[2,3] the case of circular orthotropic places has not received much attention.

Method of Solution

Assuming the classical small-deflection theory, neglecting rotatory inertia, and taking the damping forces to be proportional to the velocity, the motion of the orthotropic plate is governed by the following differential equation:

$$\nabla_0{}^4 w + kw_{,t} + \rho h w_{,tt} = 0 \tag{1}$$

The solution for w can be taken as

$$w(x,y,t) = f(x,y)\phi(t) \tag{2}$$

where $f(x,y)$ is the characteristic function chosen to satisfy the boundary conditions. So

$$\delta w = f(x,y)\delta\phi(t) \tag{3}$$

For a virtual displacement δw, the criterion according to Galerkin can be stated as

$$\iint L(w)\delta w\, dx\, dy = 0 \tag{4}$$

where $L(w)$ stands for the expression on left side of Eq. (1), and the double integral is taken over the area of the plate.

Taking

$$\phi(t) = e^{-\alpha t}\cos\omega t \tag{5}$$

and substituting Eqs. (2) and (3) in Eq. (1) gives, for all t,

$$\nabla_0{}^4 f(x,y) - \lambda^2 f(x,y) = 0 \tag{6}$$

where λ^2 is given by

$$\omega^2 = (\lambda^2/\rho h) - (k/2\rho h)^2 \tag{7}$$

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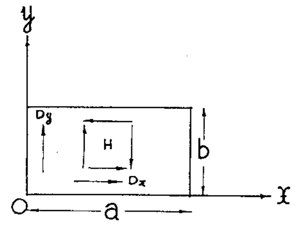
Fig. 1 Geometry of the plate



Now Eq. (4) takes the form

$$\iint (\nabla_0{}^4 f - \lambda^2 f) f(x,y) dx\, dy = 0 \quad (8)$$

In polar coordinates, Eq. (8) can be put as

$$\iint (\nabla_1{}^4 \psi - \lambda^2 \psi)\psi(r,\theta) r\, dr\, d\theta = 0 \quad (9)$$

where $\nabla_1{}^4$ can be obtained from $\nabla_0{}^4$ by means of the relations $x = r \cos\theta$, $y = r \sin\theta$, and $\psi(r,\theta)$ is a function satisfying the boundary conditions. For a circular plate, ψ is independent of θ, and so Eq. (9) becomes

$$\int_0^c (\nabla_{11}{}^4 \psi - \lambda^2 \psi)\psi(r)\, rdr = 0 \quad (10)$$

where λ^2 can be determined from Eq. (8) or (10), as the case may be, by choosing $f(x,y)$ or $\psi(r)$ suitably.

Rectangular Plates

Case 1

For a plate (Fig. 1) simply supported on all its sides, $f(x,y)$ for the fundamental mode is given by

$$f(x,y) = A_1 \sin(\pi x/a) \sin(\pi y/b) \quad (11)$$

On substituting in Eq. (8),

$$\lambda^2 = \pi^4[(D_x/a^4) + (2H/a^2b^2) + (D_y/b^4)] \quad (12)$$

a well-known result.[2, 3]

Case 2

For a plate simply supported along $y = 0$, $y = b$ and clamped along $x = 0$, $x = a$,

$$f(x,y) = A_2\, [\sin(\pi y/b)]\, x^2(x - a)^2 \quad (13)$$

and

$$\lambda^2 = 504(D_x/a^4) + (24\pi^2 H/a^2b^2) + (\pi^4 D_y/b^4) \quad (14)$$

as against almost the same value given in Ref. 2.

Case 3

For a plate clamped along its sides,

$$f(x,y) = A_3 x^2(x - a)^2\, y^2(y - b)^2 \quad (15)$$

$$\lambda^2 = 504(D_x/a^4) + 288(H/a^2b^2) + 504(D_y/b^4) \quad (16)$$

and this same value was obtained in Ref. 2. by another method.

Circular Plates

Case 1

For a plate clamped along its boundary (Fig. 2),

$$\psi(r) = A_4[1 - 2(r^2/c^2) + (r^4/c^4)] \quad (17)$$

$$\lambda^2 = (40/c^4)[D_x + (2H/3) + D_y] \quad (18)$$

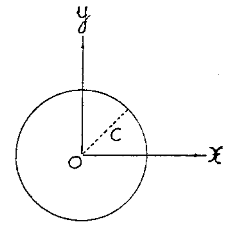
For isotropy in this case, $\lambda^2 = 106.7D/c^4$, comparing well with the exact value $\lambda^2 = 104.2\, D/c^4$ of Ref. 4.

Case 2

For a plate simply supported along its boundary,

$$\psi(r) = A_5\left[1 - \left(\frac{6 + 2\nu}{5 + \nu}\right)\frac{r^2}{c^2} + \left(\frac{1 + \nu}{5 + \nu}\right)\frac{r^4}{c^4}\right] \quad (19)$$

Fig. 2 Geometry of the plate



Taking $\nu = 0.3$, Eq. (10) gives

$$\lambda^2 = (1/c^4)[9.24D_x + 5.62H + 9.24D_y] \quad (20)$$

which in the case of isotropy becomes $\lambda^2 = 24.1\, D/c^4$, almost coinciding with the exact value $\lambda^2 = 23.60\, D/c^4$ reported in Ref. 4.

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Computation of Kinetic Constants from Single-Pulse Shock Tube Data

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Nomenclature

A	= concentration of reactant, moles per unit volume
c	= concentration of reactant, moles per unit mass of total gas
ρ	= total gas density, mass per unit volume
M	= total gas concentration, moles per unit volume
k	= kinetic rate constant
$\mathfrak{A}$	= Arrhenius pre-exponential factor in k
E	= Arrhenius activation energy
R	= molar gas constant
T	= absolute temperature
F_5	= fraction of reactant decomposed or reacted in region 5
F_c	= fraction of reactant decomposed or reacted in region 5 plus rarefaction
Region 5	= region between arrival of reflected shock and arrival of rarefaction
Region r	= region between arrival of rarefaction and achievement of effective cutoff of reaction
c_p	= constant-pressure specific heat of total gas mixture
Q	= heat release per mole of reactant decomposed

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